

A New Variable Ordering for In-processing Bounded Variable Elimination in SAT Solvers

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OUTLINE

1. Background
2. Resolution Rule
3. ESA and Experiments

Background

WHAT'S SAT?

- variable : $x_1, x_2, x_3, x_i \in \{0, 1\}$
- literal : $x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3$
- clause : $(\bar{x}_1), (x_1 \vee x_2 \vee x_3), (\bar{x}_2), (x_2 \vee \bar{x}_3)$
- SAT formula : $(\bar{x}_1) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2) \wedge (x_2 \vee \bar{x}_3)$
short : $\bar{x}_1 \wedge x_1 x_2 x_3 \wedge \bar{x}_2 \wedge x_2 \bar{x}_3$

$$(\bar{x}_1) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2) \wedge (x_2 \vee \bar{x}_3)$$

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short : $\bar{x}_1 \wedge x_1 x_2 x_3 \wedge \bar{x}_2 \wedge x_2 \bar{x}_3$

solving SAT: trying to find an assignment that satisfies all clauses in the formula.

$$(\bar{x}_1) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2) \wedge (x_2 \vee \bar{x}_3)$$

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$$(\bar{x}_1) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2) \wedge (x_2 \vee \bar{x}_3)$$

Unsatisfiable.

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solving SAT: trying to find an assignment that satisfies all clauses in the formula.

$$(\bar{x}_1) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2) \wedge (x_2 \vee \bar{x}_3)$$

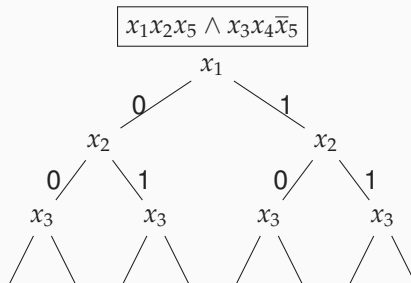
Unsatisfiable.

$$(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2) \wedge (x_2 \vee \bar{x}_3)$$

Satisfiable.

HOW TO SOLVE?

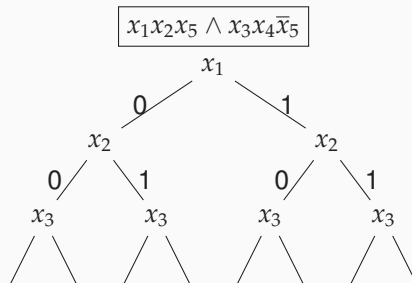
- Exhaustive Method



For a SAT formula with n variables, we need to check 2^n assignments at the worst case.

HOW TO SOLVE?

- Exhaustive Method



For a SAT formula with n variables, we need to check 2^n assignments at the worst case.

That's too much!

HOW TO SOLVE?

- Resolution Rule

$$\begin{array}{c} (x \vee l_1 \vee l_2 \vee \dots \vee l_n) \\ (\bar{x} \vee l'_1 \vee l'_2 \vee \dots \vee l'_m) \\ \Downarrow \\ (l_1 \vee l_2 \vee \dots \vee l_n \vee l'_1 \vee l'_2 \vee \dots \vee l'_m) \end{array}$$

HOW TO SOLVE?

- Resolution Rule

$$\begin{array}{c} (x \vee l_1 \vee l_2 \vee \dots \vee l_n) \\ (\bar{x} \vee l'_1 \vee l'_2 \vee \dots \vee l'_m) \\ \Downarrow \\ (l_1 \vee l_2 \vee \dots \vee l_n \vee l'_1 \vee l'_2 \vee \dots \vee l'_m) \end{array}$$

Which variable should be resolved on first?

$$xx_1x_2 \wedge \bar{x}x_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$

Resolved on x_1 : we get $xx_2x_3, \bar{x}x_2x_3$

Resolved on x_2 : we get $xx_1x_4, \bar{x}x_1x_4$

Resolved on x_3 : we get $\bar{x}_1x_5x, \bar{x}_1x_6\bar{x}$

...

RESOLUTION DRIVEN BY VARIABLE ELIMINATION

$$xx_1x_2 \wedge \bar{x}x_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$

RESOLUTION DRIVEN BY VARIABLE ELIMINATION

$$x\bar{x}_1x_2 \wedge \bar{x}\bar{x}_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$

$$\bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge x\bar{x}_2x_3 \wedge \bar{x}\bar{x}_2x_3$$

RESOLUTION DRIVEN BY VARIABLE ELIMINATION

$$x\bar{x}_1x_2 \wedge \bar{x}\bar{x}_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$

$$\bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge x\bar{x}_2x_3 \wedge \bar{x}\bar{x}_2x_3$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge x\bar{x}_3x_4 \wedge \bar{x}\bar{x}_3x_4$$

RESOLUTION DRIVEN BY VARIABLE ELIMINATION

$$x\bar{x}_1x_2 \wedge \bar{x}\bar{x}_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$

$$\bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge x\bar{x}_2x_3 \wedge \bar{x}\bar{x}_2x_3$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge x\bar{x}_3x_4 \wedge \bar{x}\bar{x}_3x_4$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_4x_5x \wedge \bar{x}_4x_6\bar{x} \wedge x_4x_5x \wedge x_4x_6\bar{x}$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge x_5x \wedge x_6\bar{x}$$

$$\bar{x}_6\bar{x} \wedge x_6\bar{x} \wedge x$$

$$x \wedge \bar{x}$$

☐ Conflict happens.

Conflict: Clauses x and \bar{x} can't be satisfied at the same time, a **conflict** happens.

RESOLUTION DRIVEN BY VARIABLE ELIMINATION

$$x\bar{x}_1x_2 \wedge \bar{x}\bar{x}_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$

$$\bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge x\bar{x}_2x_3 \wedge \bar{x}\bar{x}_2x_3$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge x\bar{x}_3x_4 \wedge \bar{x}\bar{x}_3x_4$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_4x_5x \wedge \bar{x}_4x_6\bar{x} \wedge x\bar{x}_4x_5x \wedge x\bar{x}_4x_6\bar{x}$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge x\bar{x}_5x \wedge x\bar{x}_6\bar{x}$$

$$\bar{x}_6\bar{x} \wedge x\bar{x}_6\bar{x} \wedge x$$

$$x \wedge \bar{x}$$

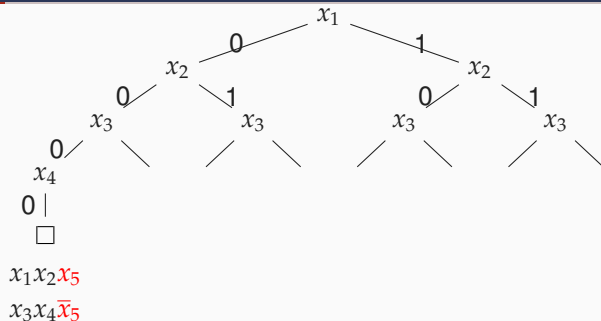
☐ Conflict happens.

Conflict: Clauses x and \bar{x} can't be satisfied at the same time, a **conflict** happens.

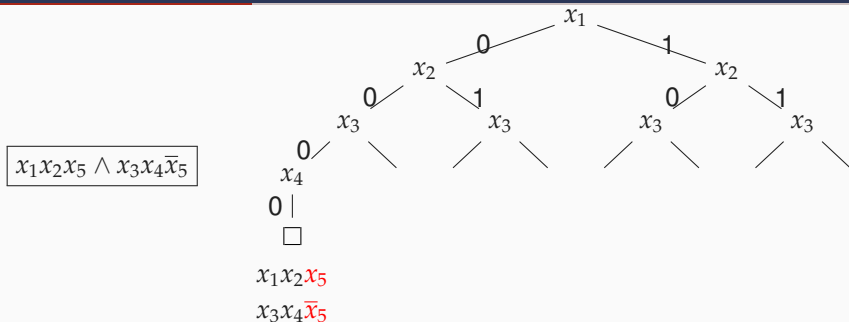
If the original formula is unsatisfiable, we will absolutely get an empty clauses by doing resolution[Robinson, 1965].

RESOLUTION DRIVEN BY CONFLICT

$$x_1x_2x_5 \wedge x_3x_4\bar{x}_5$$



RESOLUTION DRIVEN BY CONFLICT



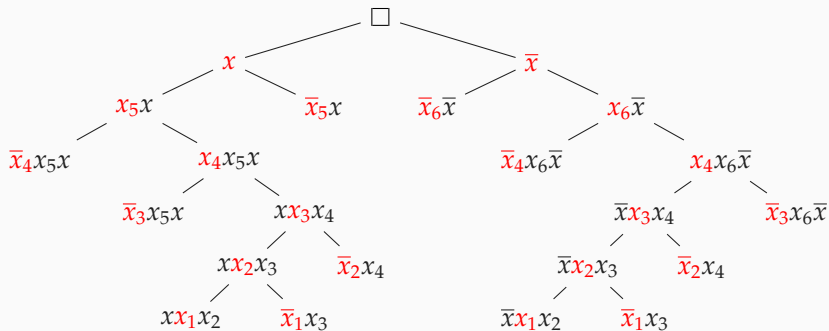
- **Conflict:** Clauses $x_1 x_2 x_5$ and $x_3 x_4 \bar{x}_5$ can't be satisfied at the same time, a **conflict** happens.
- **Conflict-Driven Clause Learning(CDCL):** By doing resolution with the two clauses $x_1 x_2 x_5$ and $x_3 x_4 \bar{x}_5$, a new clause $x_1 x_2 x_3 x_4$ is produced and added to the formula. The new clause is **learnt clause** and this process called **Conflict-Driven Clause Learning(CDCL)**.

Resolution Rule

REGULAR RESOLUTION

Variables can be resolved on **no more than once** in any path of resolution tree.

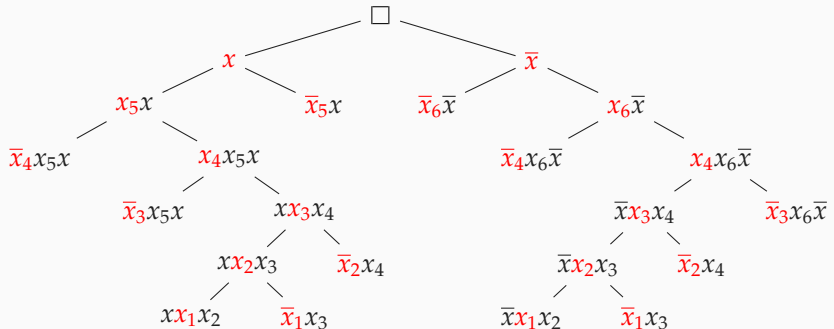
$$xx_1x_2 \wedge \bar{x}x_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$



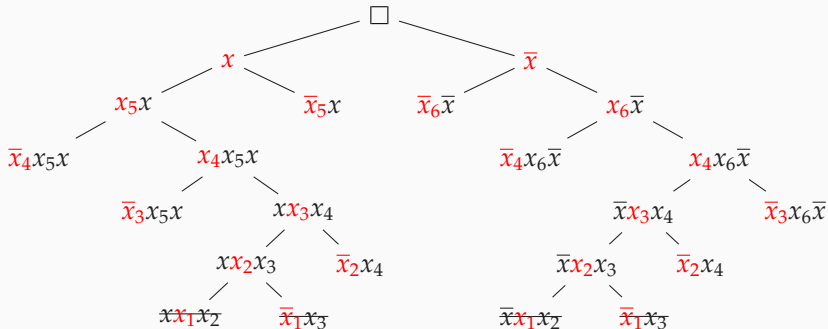
11 resolution steps

REGULAR RESOLUTION

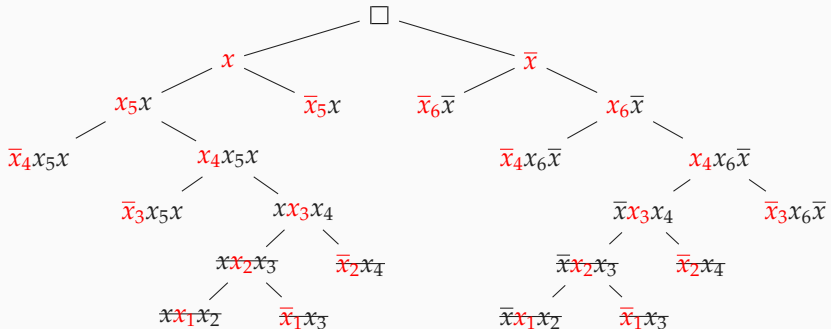
$$xx_1x_2 \wedge \bar{x}x_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$



REGULAR RESOLUTION

$$xx_1x_2 \wedge \bar{x}x_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$
$$\bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge xx_2x_3 \wedge \bar{x}x_2x_3$$


REGULAR RESOLUTION

$$xx_1x_2 \wedge \bar{x}x_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$
$$\bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge xx_2x_3 \wedge \bar{x}x_2x_3$$
$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge xx_3x_4 \wedge \bar{x}x_3x_4$$


REGULAR RESOLUTION

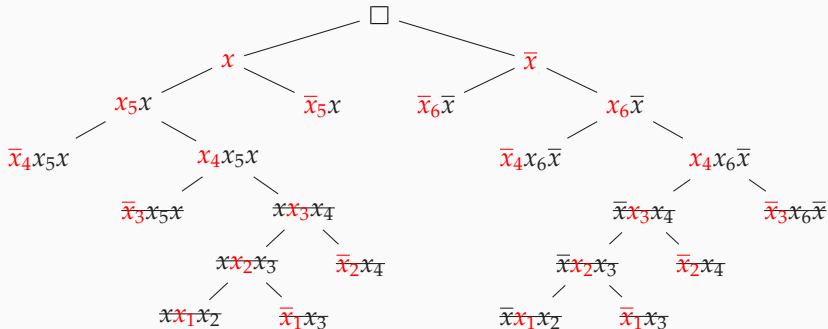
$$xx_1x_2 \wedge \bar{x}x_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$

$$\bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge xx_2x_3 \wedge \bar{x}x_2x_3$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x} \wedge xx_3x_4 \wedge \bar{x}x_3x_4$$

$$\bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_4x_5x \wedge \bar{x}_4x_6\bar{x} \wedge x_4x_5x \wedge x_4x_6\bar{x}$$

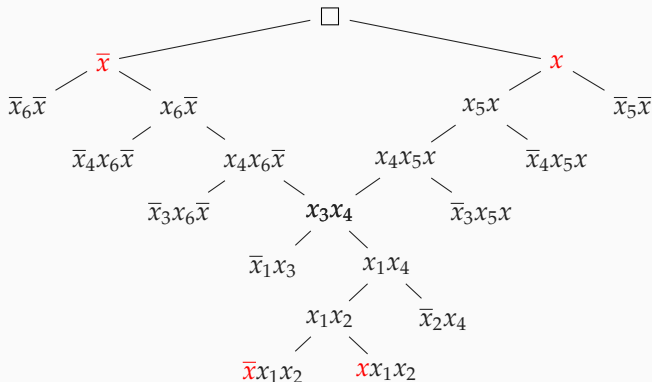
Regular resolution is used to **eliminate variables and its clauses** in formula. The formula is **simplified**.



GENERAL(UNRESTRICT) RESOLUTION

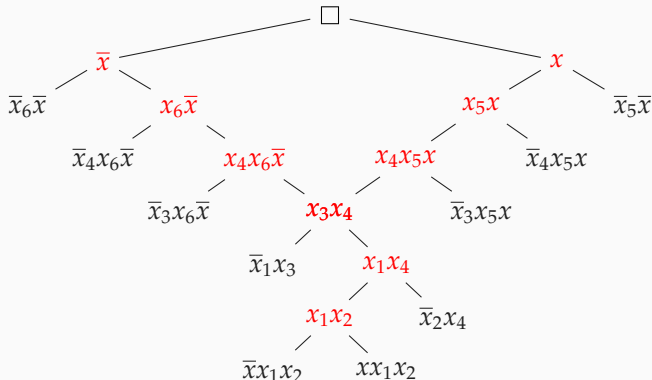
Variables can be resolved on **unlimited times**.

$$xx_1x_2 \wedge \bar{x}x_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$



10 resolution steps

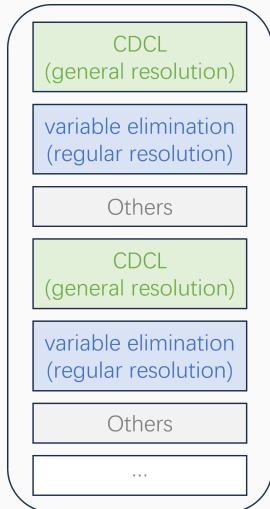
General resolution is exponentially stronger than regular resolution. [Huang and Yu, 1987; Goerdts, 1993]

$$x_1x_2 \wedge \bar{x}_1x_2 \wedge \bar{x}_1x_3 \wedge \bar{x}_2x_4 \wedge \bar{x}_5x \wedge \bar{x}_6\bar{x} \wedge \bar{x}_3x_5x \wedge \bar{x}_4x_5x \wedge \bar{x}_3x_6\bar{x} \wedge \bar{x}_4x_6\bar{x}$$


10

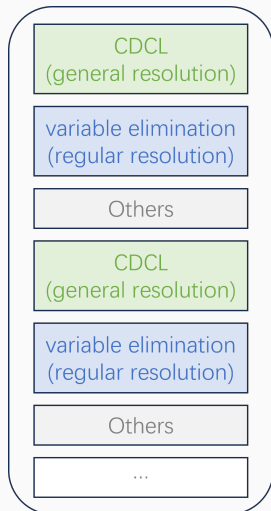
ESA and Experiments

RESOLUTION RULE IN SAT SOLVING



Cadical, Kissat, AMSAT

RESOLUTION RULE IN SAT SOLVING

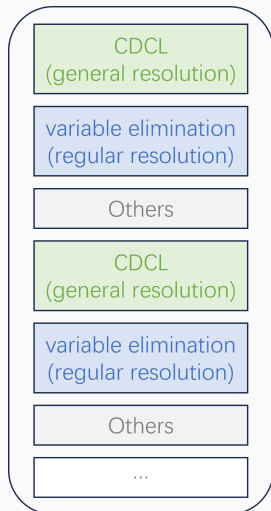


Cadical, Kissat, AMSAT

Which variable is eliminated first in Variable Elimination?

$$\begin{array}{ll} (x, l_1), (x, l_2) & \Rightarrow (l_1, l_3), (l_1, l_4) \\ (\bar{x}, l_3), (\bar{x}, l_4) & (l_2, l_3), (l_2, l_4) \\ (y, l_1), (y, l_2), (y, l_3) & (l_1, l_4), (l_1, l_5), (l_1, l_6) \\ (\bar{y}, l_4), (\bar{y}, l_5), (\bar{y}, l_6) & (l_2, l_4), (l_2, l_5), (l_2, l_6) \\ & (l_3, l_4), (l_3, l_5), (l_3, l_6) \end{array}$$

RESOLUTION RULE IN SAT SOLVING

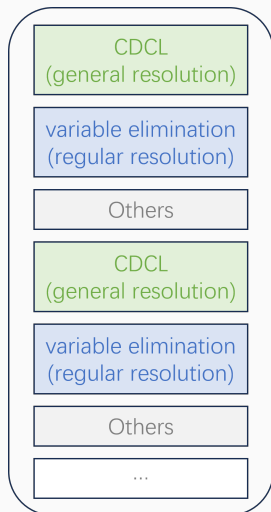


Cadical, Kissat, AMSAT

Which variable is eliminated first in Variable Elimination?

Variables **occur less often** in clauses.

RESOLUTION RULE IN SAT SOLVING

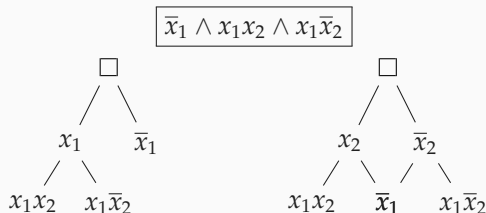


Cadical, Kissat, AMSAT

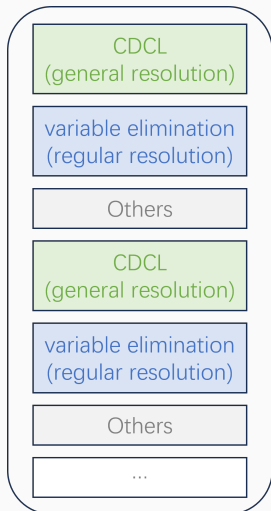
Which variable is eliminated first in Variable Elimination?

Variables **occur less often** in clauses.

Which variable is assigned first in CDCL?



RESOLUTION RULE IN SAT SOLVING



Cadical, Kissat, AMSAT

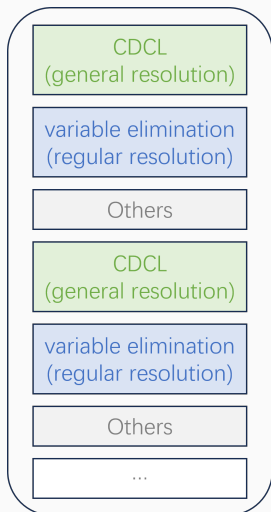
Which variable is eliminated first in Variable Elimination?

Variables **occur less often** in clauses.

Which variable is assigned first in CDCL?

Variables cause conflicts quickly. In implementation, they are variables occurs frequently in recent conflicts. This frequency is measured by a score called **activity**.

VARIABLES ELIMINATED SCHEDULED BY ACTIVITY



Cadical, Kissat, AMSAT

How to make variable elimination helpful to CDCL?

Eliminate variables with low activity, instead of variables occur in a small number of clauses.

- Eliminating those trivial variables simplifies formula, and helps the solver focus on searching in core fields.
- Keeping high activity variables also keeps their learnt clauses, those information is also helpful.

RESULTS

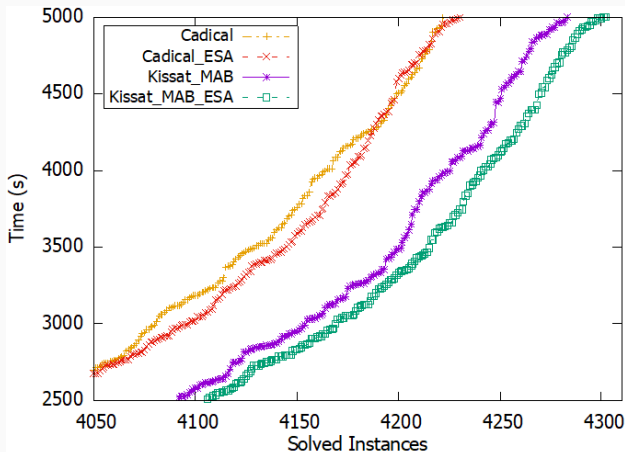


Figure 1: Cactus plots of different solvers. Each point (N, T) in a curve indicates the number N of instances solved within T secs by the corresponding solver.

RESULTS

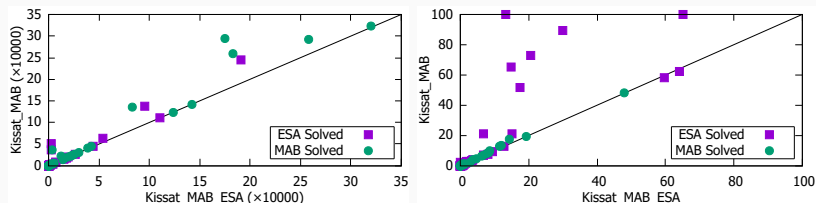


Figure 2: Scatter plot of the number of eliminated variables (left), and scatter plot of the number of learnt clauses lost per variable (right) by the 1st BVE of Kissat_MAB_ESA and Kissat_MAB.

The ESA ordering eliminates fewer variables, and loses fewer learnt clauses than Kissat_MAB when eliminating variables.

RESULTS

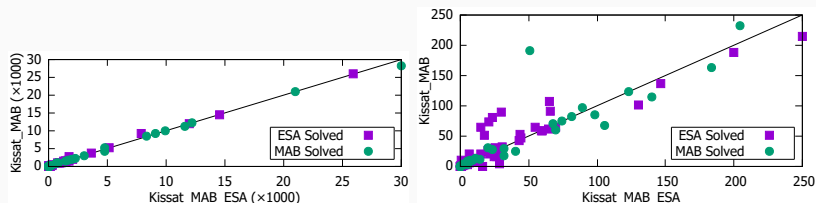


Figure 3: Scatter plot of the average number of eliminated variables per BVE call (left), and scatter plot of the average number of learnt clauses removed per variable among the first k BVE calls (right).

Kissat_MAB and Kissat_MAB_ESA eliminate roughly the same average number of variables per BVE call.

Removing learnt clauses will hurt the performance of solvers.

ESA APPLICATION IN COMPETITIONS

Anniversary Sequential Track

Winning Solvers

| | Solver | Authors | PAR-2 | Solved |
|---|---------------------|--|---------|--------|
| 1 | Kissat-MAB-ESA | Shuolin Li, Jordi Coll, Chu-Min Li, Mao Luo, Djamel Habet, and Felip Manyà | 2806.15 | 4029 |
| 2 | Kissat-sc2022-bulky | Armin Biere and Mathias Fleury | 2810.70 | 4038 |
| 3 | eKissat-MAB-gb-db | Md Solimul Chowdhury | 2829.88 | 4015 |

Anniversary Sequential Track UNSAT

Winning Solvers

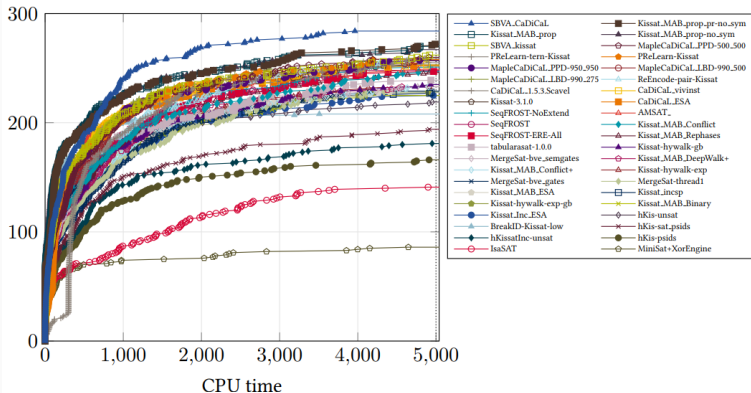
| | Solver | Authors | PAR-2 | Solved |
|---|---------------------|--|---------|--------|
| 1 | hKis-unsat | Rodrigue Konan Tchinda and Clementin Tayou Djamegni | 1071.99 | 2080 |
| 2 | Kissat-MAB-ESA | Shuolin Li, Jordi Coll, Chu-Min Li, Mao Luo, Djamel Habet, and Felip Manyà | 1093.82 | 2078 |
| 3 | Kissat-sc2022-bulky | Armin Biere and Mathias Fleury | 1115.26 | 2076 |

Kissat_MAB_ESA won the 1st in Anniversary Track and 2nd in Anniversary Unsat Track in SAT competition 2022.

Exploiting the difference between general resolution and regular resolution is useful.

ESA APPLICATION IN OTHER SOLVERS

AMSAT(Amiens Marseille SAT Solver) was submitted to SAT Competition 2023 and it outperformed all other Maple-based solvers. Variables activity is also used to choose learnt clauses to be deleted.



Notice most of these solvers are Cadical/Kissat based.

ESA APPLICATION IN OTHER SOLVERS

MaxCDCL equipped with ESA was submitted to MaxSAT Evolution 2023 and it ranked **second** in unweighted track.

Variables activity is also used in choosing learnt clauses to be deleted, learnt clauses vivification and failed literals detection.

Unweighted: 572 instances

| Solver | #Solved | Time (Avg) |
|-----------------------------|---------|------------|
| EvalMaxSAT (SCIP) | 433 | 326.11 |
| MaxCDCL (SCIP600, MaxHS900) | 430 | 244.12 |
| CASHWMAXSAT-CorePlus (SCIP) | 428 | 401.84 |